

Hamiltonian and Thermodynamic Modeling of Quantum Turbulence

Miroslav Grmela

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Abstract The state variables in the novel model introduced in this paper are the fields playing this role in the classical Landau-Tisza model and additional fields of mass, entropy (or temperature), superfluid velocity, and gradient of the superfluid velocity, all depending on the position vector and another three dimensional vector labeling the scale, describing the small-scale structure developed in ^4He superfluid experiencing turbulent motion. The fluxes of mass, momentum, energy, and entropy in the position space as well as the fluxes of energy and entropy in scales, appear in the time evolution equations as explicit functions of the state variables and of their conjugates. The fundamental thermodynamic relation relating the fields to their conjugates is left in this paper undetermined. The GENERIC structure of the equations serves two purposes: (i) it guarantees that solutions to the governing equations, independently of the choice of the fundamental thermodynamic relation, agree with the observed compatibility with thermodynamics, and (ii) it is used as a guide in the construction of the novel model.

Keywords Superfluid helium turbulence · Nonequilibrium thermodynamics · GENERIC

1 Introduction

Fluids with an internal structure that evolves in time on a scale that is comparable with the scale on which the hydrodynamic fields evolve are called complex fluids. The internal structure may be the structure of suspended objects (e.g. large macromolecules or particles) or also objects created in the flow (e.g. vortex filaments created in turbulent flows). Experimental investigation can be directed towards observations of the overall flow behavior (e.g. observations of responses to an imposed overall flow made in rheological measurements), direct observations of the internal structure, or simultaneous observations of both the overall flow and the internal structure. The point of departure of theoretical investigations of

M. Grmela (✉)
Ecole Polytechnique de Montreal, C.P. 6079, suc. Centre-ville, Montreal H3C 3A7, Quebec, Canada
e-mail: miroslav.grmela@polymtl.ca

complex fluids is always a description that is more microscopic than the hydrodynamic description. We shall call it a mesoscopic description. It can be a type of kinetic theory (as for example in [1] in the context of polymeric fluids) or an extended hydrodynamic theory in which hydrodynamic fields are promoted to the status of random variables (as it is done for example in the Reynolds-type modeling of turbulence) or still another type of extended hydrodynamics, namely a nonlocal extension of classical hydrodynamics obtained for example by successively differentiating its governing equations with respect to the position coordinate and assigning to the derivatives—notably to the vorticity—the status of independent state variables. Having specified the mesoscopic description, the next step is its reduction to an extended hydrodynamics consisting of equations of classical hydrodynamics coupled to equations governing the time evolution of the internal structure characterized usually by certain moments of the distribution functions used as state variables in the mesoscopic theory. A very useful strategy allowing to make extensions and subsequent reductions without making closure approximations that are usually very difficult to argue is to recognize in the classical hydrodynamic equations a mathematical structure of physical importance and then to keep it in all modifications that follow. The structure that we shall use for this purpose is an appropriate combination (called GENERIC—see Sect. 2) of the Hamiltonian structure of mechanics with the dissipative structure guaranteeing the compatibility with thermodynamics.

Our objective in this paper is to model quantum turbulence in ^4He superfluid. The role of classical hydrodynamics is thus played by the Landau-Tisza two-fluid model. We extended it by following two routes. The first one is the well known route leading to the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) theory. We shall recall it in Sect. 4 and cast it (by using [2]) into GENERIC form. The second route is new. The starting mesoscopic theory (Sect. 5) is a nonlocal extension of the Landau model made by passing from its one-point to the two-point formulation. Both the way we extend the original theory and the way we make all extensions and reductions by requiring the GENERIC structure to be preserved are novel features that we are introducing into the quantum turbulence modeling.

2 GENERIC

All time evolution equations appearing in this paper possess the structure of GENERIC.

The GENERIC equation

$$\dot{x} = LE_x + \frac{\partial \Xi}{\partial H_x} \quad (1)$$

is an abstract equation governing the time evolution of state variables x ; by \dot{x} we denote the derivative of x with respect to the time t . Equation (1) combines in an appropriate way the Hamiltonian and nondissipative time evolution (governed by the first term on the right hand side of (1)) with the Ginzburg-Landau-Cahn-Hilliard type dissipative time evolution (governed by the second term on the right hand side of (1)). L is a Poisson bivector transforming the gradient of energy $E(x)$ (that is a real valued function of x ; E_x is a shorthand notation for $\partial E/\partial x$) into a vector, Ξ is a dissipation potential (that is a real valued function of the gradient H_x of the eta-function $H(x)$ (a real valued function of x with the property that $H(x_{eq})$ is the entropy arising in equilibrium thermodynamics; x_{eq} is the $t \rightarrow \infty$ solution to (1)). The Poisson bivector L and the dissipation potential Ξ are required to possess the following properties:

(i)

$$\langle A_x, LB_x \rangle = \{A, B\} \tag{2}$$

is a Poisson bracket; $A(x), B(x)$ are sufficiently regular real valued function of x , $\langle \cdot, \cdot \rangle$ denotes the inner product,

(ii)

$$\Xi(0) = 0; \quad \Xi \text{ reaches its minimum at } 0; \quad \Xi \text{ is convex in a neighborhood of } 0, \tag{3}$$

and

(iii) both L and Ξ are appropriately degenerate so that

$$\begin{aligned} \dot{E} &= 0 \\ \dot{N} &= 0 \\ \dot{H} &\geq 0 \end{aligned} \tag{4}$$

holds. By $N(x)$ we denote a real valued function of x having the physical meaning of the number of moles. The “nonequilibrium entropy” $H(x)$ is called eta-function in order to avoid possible confusion with many other nonequilibrium entropies existing in the literature. The last inequality is a consequence of the degeneracy of the Poisson bracket (namely the property $\{A, H\} = 0$ for all A) and the inequality $H_x \Xi_{H_x} > 0$ that follows from (3).

The Hamiltonian structure of the nondissipative part of the time evolution has been discovered first in the context of hydrodynamics by Clebsch in [3]. Equations of the type (1) have started to appear in [4] and later in [5–9]. In the form (1) and with the name GENERIC, the abstract time evolution equation (1) has appeared first in [10, 11]. GENERIC has been then further developed in [12–15] and in a different direction in [16, 17].

The physical significance of (1) is based on the fact that solutions to (1) are guaranteed to agree with thermodynamics and mechanics.

The compatibility with mechanics is mathematically expressed by the requirement that L in (1) is a Poisson bivector.

The compatibility with thermodynamics is mathematically expressed as follows: (i) as $t \rightarrow \infty$, solutions to (1) converge to equilibrium states (denoted $x_{eq}(T, \mu)$) that are the states minimizing the mesoscopic thermodynamic potential

$$\Phi(x, T, \mu) = -H(x) + \frac{1}{T}E(x) - \frac{\mu}{T}N(x) \tag{5}$$

where T is the temperature and μ the chemical potential.

(ii) $\Phi(x_{eq}(T, \mu), T, \mu) = -\frac{PV}{T}$ is the fundamental thermodynamic relation $P = P(T, \mu)$ of classical equilibrium thermodynamics implied by the mesoscopic thermodynamic potential (5). By P we denote the thermodynamic pressure and V stands for the volume of the region in which the macroscopic system under consideration is confined. From (4) we indeed see that the mesoscopic thermodynamic potential (5) can play the role of the Lyapunov function implying $x \rightarrow x_{eq}$ as $t \rightarrow \infty$ provided it is a convex function of x .

In this paper we shall formulate three models of ^4He superfluid. The first two (appearing in Sects. 3 and 4) are known, the third (appearing in Sect. 5) is new. The governing equations of all three models are particular realizations of (1) (i.e. (1) in which x, L, Ξ take a concrete identity) and are constructed by following the same route guided by the GENERIC structure. The route has three stages: *Stage 1*: choice of the state variables x , *Stage 2*: specification of

their Hamiltonian kinematics expressed in the Poisson bivector L and their dissipative kinematics expressed in the dissipation potential Ξ , and *Stage 3*: specification of the potentials $H(x)$, $E(x)$, $N(x)$ whose gradients are the forces driving both the nondissipative and dissipative time evolution. The viewpoint that we choose to regard ^4He superfluid (in other words the chosen level of description) is expressed mathematically in the first stage. The second stage determines how the forces, that are still undetermined and will be determined only in the third stage, are combined in the vector field generating the time evolution of the state variables x chosen in the first stage. In the third stage the forces are then specified. In this paper we shall concentrate on the first two stages. The third stage is completed in the first two models (in Sects. 3 and 4) but it remains incomplete in the third model (in Sect. 5).

Before proceeding with this program we note that the way the physical insight is expressed mathematically by following Stage 1 then Stage 2 and finally Stage 3 is in a sense reverse of the traditional way the models are constructed. The traditional modeling is more bottom-up oriented while the modeling guided by GENERIC has the top-down character. Both approaches should be seen as complementary. The reader can clearly see the advantages and disadvantages of both approaches in Sects. 3 and 4 where the traditional derivations of the governing equations are well known. The model introduced in Sect. 5 with the GENERIC method is new. The bottom-up way of thinking is needed to fill up the framework (i.e. to complete the third stage in the GENERIC modeling) constructed in Sect. 5.

3 Landau-Tisza Model

The nondissipative part of the Landau-Tisza two-fluid model has been cast into the Hamiltonian form by Holm and Kupersmidt in [18]. The GENERIC extension to the complete two-fluid model has appeared in [19]. We shall now recall it.

3.1 State Variables x

The state variables used in [18] and [19] are:

$$x^{(1)} = (\rho(\mathbf{r}), \mathbf{u}(\mathbf{r}), \eta(\mathbf{r}), \mathbf{v}_s(\mathbf{r})) \tag{6}$$

$\mathbf{r} \in \mathbb{R}^3$, where ρ is the mass per unit volume, \mathbf{u} is the overall momentum per unit volume, η is the entropy per unit volume, and \mathbf{v}_s the velocity of the superfluid. The first three fields in (6) are the classical hydrodynamic fields and the fourth field is specific to the superfluid. The physical interpretation of (6) immediately leads to the specification of $N(x)$ and $H(x)$ appearing in (1):

$$N(x^{(1)}) = \frac{1}{M_{mol}} \int d\mathbf{r} \rho(\mathbf{r}) \tag{7}$$

where M_{mol} is the molecular weight and

$$H(x^{(1)}) = \int d\mathbf{r} \eta(\mathbf{r}) \tag{8}$$

The third function

$$E(x^{(1)}) = \int d\mathbf{r} e(\rho, \mathbf{u}, \eta, \mathbf{v}_s; \mathbf{r}) \tag{9}$$

will be considered as the fundamental thermodynamic relation of the Landau two-fluid model. Following the spirit of thermodynamics, we shall try to postpone its specification as much as possible (to Sect. 3.5). The focus is put in this paper on the kinematics and the dissipation potential.

Alternatively, we shall also use

$$\xi^{(1)} = (\rho(\mathbf{r}), \mathbf{u}(\mathbf{r}), e(\mathbf{r}), \mathbf{v}_s(\mathbf{r})) \tag{10}$$

as the state variables. The fields ρ, \mathbf{u} and \mathbf{v}_s are the same as in (6) and e is the local energy appearing in the fundamental thermodynamic relation (9). There is a one-to-one relation between (6) and (10) since the local temperature θ given in (33) is assumed to be always positive. By using the terminology of [21], (6) are referred as state variables in the energy representation and (10) as state variables in the entropy representation. With (10) the functions $N(\xi^{(1)})$ is the same as (7) and $E(\xi^{(1)})$ is given by

$$E(\xi^{(1)}) = \int d\mathbf{r} e(\mathbf{r}) \tag{11}$$

The fundamental thermodynamic relation takes now the form

$$H = \int d\mathbf{r} \eta(\rho, \mathbf{u}, e, \mathbf{v}_s; \mathbf{r}) \tag{12}$$

We recall that the conjugate state variables of (6) and (10) are related by

$$E_\eta = (H_e)^{-1}; \quad E_u = -H_u/H_e; \quad E_{v_s} = -H_{v_s}/H_e \tag{13}$$

Following nonequilibrium thermodynamics, we equip the state space with an additional structure. We introduce a transformation (called a parity transformation)

$$\mathbf{I} : (\rho, \mathbf{u}, \eta, \mathbf{v}_s) \mapsto (\rho, -\mathbf{u}, \eta, -\mathbf{v}_s) \tag{14}$$

for the choice (6) and

$$\mathbf{I} : (\rho, \mathbf{u}, e, \mathbf{v}_s) \mapsto (\rho, -\mathbf{u}, e, -\mathbf{v}_s) \tag{15}$$

for the choice (10) We note that the parity transformation \mathbf{I} is idempotent (i.e. $\mathbf{I} \circ \mathbf{I} =$ identity mapping) and that it splits the state variables into two classes (we use the terminology introduced in [20]): the α -variables $(\rho, 0, \eta, 0)$ that are invariant with respect to \mathbf{I} , and β -variables $(0, \mathbf{u}, 0, \mathbf{v}_s)$ that change the sign if \mathbf{I} is applied on them. The involution \mathbf{I} induces naturally a mapping in the space of the vector fields (i.e. in the right hand sides of the time evolution equations). The time evolution equation is called time reversible if it is invariant with respect to the application of $t \rightarrow -t$ and $x \rightarrow \mathbf{I}(x)$.

3.2 Kinematics L

Now we turn our attention to the Hamiltonian kinematics of the fields (6). Following [18], the Poisson bracket expressing it is given by

$$\begin{aligned} \{A, B\}^{(1)} = & \int d\mathbf{r} [u_k(\partial_i(A_{u_k})B_{u_i} - \partial_i(B_{u_k})A_{u_i}) + \rho(\partial_i(A_\rho)B_{u_i} - \partial_i(B_\rho)A_{u_i}) \\ & + \eta(\partial_i(A_\eta)B_{u_i} - \partial_i(B_\eta)A_{u_i}) + v_{sk}(\partial_i(A_{u_k})B_{v_{si}} - \partial_i(B_{u_k})A_{v_{si}}) \\ & + \partial_k(v_{si})(A_{u_k}B_{v_{si}} - B_{u_k}A_{v_{si}}) - (A_\rho\partial_k(B_{v_{sk}}) - B_\rho\partial_k(A_{v_{sk}}))] \end{aligned} \tag{16}$$

Kinematics of the motion, that is seen as a sequence of transformations $\mathbf{R}^3 \rightarrow \mathbf{R}^3$, is expressed in the first line. The overall momentum \mathbf{u} is regarded as an element of the dual of the Lie algebra corresponding to the Lie group of transformations $\mathbf{R}^3 \rightarrow \mathbf{R}^3$. The second and the third lines express passive advection of the two scalar fields ρ and η . The fourth and the fifth lines express passive advection of the field \mathbf{v}_s regarded as a dual element to 2-forms on \mathbf{R}^3 . The sixth line is a two-cocycle compatible with the previous five lines.

It is easy to verify that both N and H introduced in (7) and (8) are Casimir functions for the Poisson bracket (16), i.e.

$$\{A, N\}^{(1)} = \{A, H\}^{(1)} = 0 \quad \text{for all } A \tag{17}$$

The Poisson bracket expressing kinematics of the state variables (10) is obtained from the Poisson bracket (16) by applying on it the one-to-one transformation between (6) and (10). We shall not write it down explicitly.

3.3 Dissipation potential Ξ

Next, we turn to the dissipation kinematics. In order that (4) holds, the degeneracy (17) of the Hamiltonian part of (1) has to be supplemented by the degeneracy of the dissipative part of (1) guaranteeing the conservation of the number of moles N and the energy E . It will be easier to construct the dissipative part with the required degeneracy in the entropy representation (i.e. with the state variables (10)).

The number of moles N remains the same as in (7) but (9) is replaced by (12).

As it is customary in the analysis of the dissipative time evolution in the context of classical nonequilibrium thermodynamics, we first identify the thermodynamic forces (denoted by the symbol \mathbf{X}) driving the system under consideration to equilibrium. We specify them as specific functions of gradients of still unspecified potentials H, E, N . We choose them to be

$$\begin{aligned} X_i^{(e)} &= \partial_i(H_e) \\ X_{ij}^{(u)} &= \frac{1}{2} \left(\partial_j \left(\frac{H_{u_i}}{H_e} \right) + \partial_i \left(\frac{H_{u_j}}{H_e} \right) \right) \\ X_i^{(v_s)} &= H_{v_{si}} \end{aligned} \tag{18}$$

The first force $\mathbf{X}^{(e)}$ is the classical Fourier force driving the Fourier heat conduction. We shall also use the traditional notation: $H_e = \frac{1}{\theta}$, where θ is the local temperature.

The second force $\mathbf{X}^{(u)}$ is also a classical force. It is the Navier-Stokes force. Indeed, as it is clear from (13), $-\frac{H_u}{H_e} = E_u$. In the classical fluid E_u is the velocity field since the only term in the energy that depends on \mathbf{u} is the kinetic energy $\int d\mathbf{r} \frac{\mathbf{u}^2}{2\rho}$, where ρ is the mass density. In the Landau-Tisza model of ^4He fluid, E_u is also a velocity field but only of the normal component (see (33)) denoted \mathbf{v}_n . The force $\mathbf{X}^{(u)}$ is thus indeed the classical Navier-Stokes force (i.e. the symmetric gradient of the velocity) of the normal component of the ^4He fluid.

The third force $\mathbf{X}^{(v_s)}$ is new. As we shall see below in (33), the thermodynamics force $X_i^{(v_s)} = -\rho_s(v_{si} - v_{ni})/\theta$ is a force that drives the ^4He fluid to states at which the normal fluid and the superfluid have the same velocity.

Next, we need to introduce a real valued function Ξ of the thermodynamic forces $\mathbf{X}^{(e)}, \mathbf{X}^{(u)}, \mathbf{X}^{(v_s)}$ satisfying (3) and (4), called dissipation potential In order to narrow down

the choice, we limit ourselves to states that are not too far from equilibrium (more precisely, to states at which we can neglect terms proportional to $(X^{(e)})^k, (X^{(u)})^k, (X^{(v_s)})^k; k \geq 3$) and introduce the following quadratic dissipation potential

$$\Xi = \Xi^{(e)} + \Xi^{(u)} + \Xi^{(v_s)} \tag{19}$$

where

$$\Xi^{(e)} = \frac{1}{2} \int d\mathbf{r} X_i^{(e)} \Lambda^{(e)} X_i^{(e)} \tag{20}$$

$$\Xi^{(u)} = \frac{1}{2} \int d\mathbf{r} X_{ij}^{(u)} \Lambda^{(u)} X_{ij}^{(u)} \tag{21}$$

$$\Xi^{(v_s)} = \frac{1}{2} \int d\mathbf{r} X_i^{(v_s)} \Lambda_{ij}^{(v_s)} X_j^{(v_s)} \tag{22}$$

$\Lambda^{(e)}, \Lambda^{(u)},$ and $\Lambda^{(v_s)}$ are material parameters. We note that if $\Lambda^{(e)} > 0; \Lambda^{(u)} > 0$ and $\Lambda^{(v_s)}$ is positive definite then the dissipation potential (19) clearly satisfies all the required properties (3) and (4). All the kinetic coefficients $\Lambda^{(e)}, \Lambda^{(u)},$ and $\Lambda^{(v_s)}$ can depend on the state variables. An interesting example of $\Lambda^{(v_s)}$, discussed more detail in Sect. 4—see (42), is $\Lambda_{ij}^{(v_s)} = \Lambda^{(v_s)} \Omega_{ik} \Omega_{kj}$, where $\Lambda^{(v_s)} > 0$ and $\Omega_{ik} = \partial_i v_{sk} - \partial_k v_{si}$.

3.4 GENERIC Time Evolution Equations

By inserting (16) into (1) (or alternatively into its reformulation in terms of the state variables (10) and (19)) we arrive at the time evolution equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\partial_k J_k^{(\rho)} \\ \frac{\partial \eta}{\partial t} &= -\partial_k (\eta E_{u_k}) - \partial_k J_k^{(\eta)} + \sigma \\ \frac{\partial u_i}{\partial t} &= -\partial_k T_{ik} \\ \frac{\partial v_{si}}{\partial t} &= -\partial_i E_\rho - \partial_k (v_{si}) E_{u_k} - v_{sk} \partial_i E_{u_k} + \Xi_{X_k}^{(v_s)} \end{aligned} \tag{23}$$

and

$$\frac{\partial e}{\partial t} = -\partial_k J_k^{(e)} \tag{24}$$

The mass flux $\mathbf{J}^{(\rho)}$, the stress tensor \mathbf{T} , and the energy flux $\mathbf{J}^{(e)}$ are given by

$$\mathbf{J}_k^{(\rho)} = \rho E_{u_k} + E_{v_{sk}} \tag{25}$$

$$T_{ik} = u_i E_{u_k} + p \delta_{ik} + \tau_{ik}^{(nd)} + \tau_{ik}^{(d)} \tag{26}$$

$$\mathbf{J}_k^{(e)} = (p + e) E_{u_k} + \tau_{kj}^{(nd)} E_{u_j} + q_k^{(nd)} + \tau_{kj}^{(d)} E_{u_j} + q_k^{(d)}$$

$$\mathbf{J}_k^{(\eta)} = H_e \Xi_{X_k}^{(e)} \tag{27}$$

where

$$p = -e + \rho E_\rho + \eta E_\eta + u_k E_{u_k} \tag{28}$$

$$\tau_{ik}^{(nd)} = v_{si} E_{v_{sk}} \tag{29}$$

$$\tau_{ij}^{(d)} = \frac{1}{H_e} \Lambda^{(u)} X_{ij}^{(u)}$$

$$q_i^{(nd)} = E_\rho E_{v_{si}} \tag{30}$$

$$q_i^{(d)} = \Xi_{X_k^{(e)}} = \Lambda^{(e)} X_i^{(e)} \tag{31}$$

and the entropy production

$$\sigma = X_k^{(e)} \Xi_{X_k^{(e)}} + X_{kj}^{(u)} \Xi_{X_{kj}^{(u)}} + X_k^{(v_s)} \Xi_{X_k^{(v_s)}} \tag{32}$$

Since the dissipation potential Ξ satisfies (3), the entropy production σ is positive.

Equations (23)–(32) complete the Stages 1 and 2 (see the end of Sect. 2 where this terminology is introduced) of the GENERIC modeling. What remains is the Stage 3 in which the potentials E and H are specified. We shall go through the remaining Stage 3 in Sect. 3.5. But it is in the form (23)–(32), i.e. just after completing the first two stages of the GENERIC modeling, where we can see most clearly the advantages of the GENERIC modeling.

We note that the expressions for the hydrostatic pressure, for both the nondissipative and dissipative parts of the extra stress tensor, for the heat and the entropy fluxes, and for the entropy production have arisen together with the time evolution equations as a single consequence of the GENERIC formulation. On the contrary, in the classical derivations (see e.g. [25]) of the governing equations of the two-fluid hydrodynamics of the superfluid ^4He , the time evolution equations and expressions of the type (28), (29) have to be argued separately and their mutual compatibility thus remains unclear.

The fact that we have not committed ourselves yet to specific potentials E and H is another advantage. The governing equations (23)–(32) can still be adapted to different physical situations corresponding to different choices of the potentials E and H .

Finally, irrespectively of the choice of the potentials E and H , we know that solutions of (23)–(32) agree with one particular result of experimental observations, namely with the observed approach of externally unforced ^4He superfluid to equilibrium states at which its behavior is found to be well described by equilibrium thermodynamics.

In order to understand more the situation at which we find ourselves at this point of GENERIC modeling we make an analogy with the Gibbs formulation of equilibrium thermodynamics (as presented for example in [21]). In the Gibbs formulation we can also recognize three stages. In the first stage the state variables are chosen (the choice is universal in the classical equilibrium thermodynamics). In the second stage one of the state variables (either energy or entropy) is postulated to be a function (satisfying certain properties and called a fundamental thermodynamic relation) of the remaining state variables. In the third stage the fundamental thermodynamic relation is specified. After the second stage in the Gibbs formulation of equilibrium thermodynamics we already have significant results that are of a general nature but with experimentally verifiable consequences (e.g. the notion of the conjugate state variables and Maxwell’s relations) similarly as we have analogical results after the second stage of the GENERIC modeling (e.g. expressions for the stress tensor, heat flux, and entropy flux in terms of the state variables and their conjugates).

3.5 Conjugate State Variables

In the final stage (Stage 3) of the GENERIC modeling the potentials $E(x), H(x), N(x)$ are specified. Since, as we see in (5), these potentials determine the equilibrium fundamental thermodynamic relation, the third stage is the thermodynamic part of the GENERIC modeling. The dynamic (both nondissipative and dissipative) part of the modeling has already been completed in (23)–(32). We note that the derivatives of the potentials $E(x), H(x), N(x)$ with respect to x rather than the potentials themselves appear in these equations. In the context of thermodynamics, such derivatives are the conjugate state variables. We shall denote them by the symbol x_{conj} . We can thus regard (23)–(32) as evolutions governing the time evolution of x but involving still x_{conj} that remain undetermined functions of x . In order to make (23)–(32) self-contained, we have to specify the mapping $x \leftrightarrow x_{conj}$. The viewpoint of the time evolution in which both x and x_{conj} play the role of equal importance arises naturally in the geometrical setting in which the GENERIC time evolution becomes a continuous sequence of Legendre transformations (see more in [15]).

Before specifying x_{conj} we observe the role that they play in (23)–(32). The expression (29) and in particular then the expression (30) for the heat flux provide a physical interpretation of the conjugate of the field of the superfluid velocity $\mathbf{v}_s(\mathbf{r})$ (i.e. the field that is absent in the standard one-fluid hydrodynamics of classical fluids). We see how the conjugate of the extra field $\mathbf{v}_s(\mathbf{r})$ manifests itself in the observation of the overall flow behavior characterized by the standard hydrodynamic fields $(\rho(\mathbf{r}), \mathbf{u}(\mathbf{r}), e(\mathbf{r}))$. In addition, the relation (30) makes it possible to regard (see [22, 23]) the extension from classical to the Landau two-fluid hydrodynamics as a type of the extension of classical hydrodynamics in which the heat flux is admitted as an independent state variable (i.e. the type of extensions investigated in extended irreversible thermodynamics [24]).

Now we identify x_{conj} for which (23)–(32) become the standard governing equations of the Landau two-fluid model (see e.g. [25]):

$$\begin{aligned}
 E_\rho &= \mu - \frac{1}{2}v_n^2 \\
 E_\eta &= \theta \\
 E_{u_i} &= v_{ni} \\
 E_{v_{si}} &= \rho_s(v_{si} - v_{ni}) \\
 \rho &= \rho_s + \rho_n
 \end{aligned}
 \tag{33}$$

where $\theta = E_\eta$ is the local temperature $\mu = E_\rho$ is the local chemical potential, \mathbf{v}_n is the velocity of the normal fluid, ρ_s is the mass density of the superfluid ($\mathbf{v}_s - \mathbf{v}_n$ is called a counterflow velocity and thus E_{v_s} can be called a counterflow momentum), and ρ_n is the mass density of the normal fluid. The superfluid and the normal fluid are the two fluids composing the ^4He superfluid. With (33) the mass flux takes the form

$$\mathbf{J}_k^{(\rho)} = \rho_n v_{nk} + \rho_s v_{sk}
 \tag{34}$$

and the last equation in (23)

$$\frac{\partial v_{si}}{\partial t} = -\partial_i \left(\mu - \frac{1}{2}v_n^2 + v_{sk}v_{nk} \right) + (\mathbf{v}_n \times \text{curl} \mathbf{v}_s)_i - \Lambda_{ij}^{(3)} \theta^{-1} \rho_s (v_{si} - v_{ni})
 \tag{35}$$

The quantum nature of the superfluid requires $\mathbf{v}_s = \nabla\phi$, where $\phi(\mathbf{r})$ is a potential (or equivalently $\text{curl}\mathbf{v}_s = 0$). As it is seen from the nondissipative part of (35) (i.e. (35) without the last term on its right hand side), $\text{curl}\mathbf{v}_s = 0$ for $t > 0$ provided it holds for $t = 0$.

4 HVBK Model

Under some conditions (see [26, 27]), the liquid helium is seen to develop a microstructure consisting of vortex filaments. Our objective is to extend (23) to fluids involving such internal structure. We proceed in two steps. First, we formulate a more microscopic (mesoscopic) version of the Landau-Tisza two-fluid theory. In the second step, we reduce it to an extended two-fluid model consisting of (23) supplemented by equations governing the time evolution of some extra fields that have been chosen to characterize the microstructure. In order to be certain that the physics expressed in (23) is kept in the extended and reduced time evolution equations, we shall require that the GENERIC structure identified in the previous section is retained in all stages of the analysis.

The mesoscopic extension of the two-fluid model (23) that we may expect to be of some pertinence is a nonlocal extension. This type of extension can be made for example by successively differentiating (23) with respect to the position coordinate \mathbf{r} and assigning to the derivatives the status of independent state variables. This is essentially the route leading to the HVBK theory. We recall it below in this section. An alternative route will be introduced and explored in Sect. 5.

Since the equality $\text{curl}\mathbf{v}_s = 0$ holds in the quantum fluids that we have discussed in the previous section and since the vortex filaments observed experimentally in turbulent quantum fluids are the filaments of the superfluid, an additional superfluid velocity (we shall denote it by the symbol $-\mathbf{w}(\mathbf{r})$) whose vorticity can be different from zero appears as a most natural choice of the extra field characterizing the microstructure. Alternatively, following Vinen [30], we can choose the vortex length density (that is proportional to $|\text{curl}\mathbf{w}|$) as the small-scale state variable.

Having chosen the extra state variable, the next task is to extend (23) to the equations governing the time evolution of both (6) and the new field $-\mathbf{w}(\mathbf{r})$ (see [26, 27] for a review of the HVBK model) or alternatively the time evolution of (6) and the new field $|\text{curl}\mathbf{w}|$. Below, we shall limit ourselves to the HVBK model in which the field $-\mathbf{w}(\mathbf{r})$ plays the role of the extra state variable, use the Hamiltonian formulation of the nondissipative HVBK equations [2], and cast the governing equations of the HVBK model into the GENERIC form.

4.1 State Variables x

Following [2], the state variables are chosen to be

$$x^{(HVBK)} = (\rho(\mathbf{r}), \mathbf{m}(\mathbf{r}), \eta(\mathbf{r}), \mathbf{m}_s(\mathbf{r}), \mathbf{w}(\mathbf{r}), n(\mathbf{r})) \tag{36}$$

The fields $\rho(\mathbf{r}), \mathbf{m}(\mathbf{r}), \eta(\mathbf{r}), \mathbf{m}_s(\mathbf{r})$ are the same as in (6). We have only changed the notation. We use now the symbol \mathbf{m} to denote \mathbf{u} appearing in (6) and the symbol \mathbf{m}_s to denote \mathbf{v}_s appearing in (6) (i.e. $\text{curl}\mathbf{m}_s = 0$). The scalar field $n(\mathbf{r})$ is an auxiliary state variable. Later, by appropriately choosing the fundamental thermodynamic relation (see (49), it will be shown to be the same as the field ρ . The parity transformation \mathbf{I} extending (15) is

$$\mathbf{I} : (\rho, \mathbf{m}, e, \mathbf{m}_s, \mathbf{w}, n) \mapsto (\rho, -\mathbf{m}, e, -\mathbf{m}_s, -\mathbf{w}, n) \tag{37}$$

The overall momentum $\mathbf{u}(\mathbf{r})$ and the overall superfluid velocity $\mathbf{v}_s(\mathbf{r})$ are given by

$$\begin{aligned} \mathbf{u} &= \mathbf{m} - n\mathbf{w} \\ \mathbf{v}_s &= \mathbf{m}_s - \mathbf{w} \end{aligned} \tag{38}$$

We note that $\text{curl}\mathbf{v}_s = -\text{curl}\mathbf{w}$ since $\text{curl}\mathbf{m}_s = 0$.

4.2 Kinematics L

The kinematics of $x^{(HVBK)}$ proposed by Holm in [2] is expressed in the Poisson bracket

$$\{A, B\}^{(HVBK)} = \{A, B\}^{(1)} + \int d\mathbf{r} \left[(A_n \partial_j (B_{w_j}) - B_n \partial_j (A_{w_j})) - A_{w_j} \frac{\epsilon_{kjl} (\text{curl}\mathbf{w})_l}{n} B_{w_k} \right] \tag{39}$$

where ϵ is the alternating tensor, $\{A, B\}^{(1)}$ is the Poisson bracket (16) expressing kinematics of the fields $(\rho(\mathbf{r}), \mathbf{m}(\mathbf{r}), \eta(\mathbf{r}), \mathbf{m}_s(\mathbf{r}))$ and the last two lines in (39) is a Poisson bracket expressing kinematics of the fields $(\mathbf{w}(\mathbf{r}), n(\mathbf{r}))$. The kinematics of the fields $(\rho(\mathbf{r}), \mathbf{m}(\mathbf{r}), \eta(\mathbf{r}), \mathbf{m}_s(\mathbf{r}))$ is thus completely decoupled from the kinematics of the extra fields $(\mathbf{w}(\mathbf{r}), n(\mathbf{r}))$. The new and the old fields are however coupled in (38) and in the fundamental thermodynamic relation (the conjugate fields appearing on the right hand side of all the time evolution equations depend, in general, on all the fields).

4.3 Dissipation Potential Ξ

The Hall-Vinen ([28, 29]) dissipative mutual friction force is generated by the thermodynamic force

$$X_i^{(w)} = H_{w_i} \tag{40}$$

The total dissipation potential $\Xi^{(HVBK)}$ is then a sum of the dissipation potential (19) used in the classical Landau-Tisza model and the Hall-Vinen potential $\Xi^{(HV)}$:

$$\Xi^{(HVBK)} = \Xi^{(e)} + \Xi^{(u)} + \Xi^{(v_s)} + \Xi^{(HV)} \tag{41}$$

where

$$\Xi^{(HV)} = \frac{1}{2} \int d\mathbf{r} \Lambda^{(HV)}(x^{(HVBK)}) X_i^{(w)} \Omega_{ik} \Omega_{kj} X_j^{(w)} \tag{42}$$

$\Lambda^{(HV)}(x^{(HVBK)}) > 0$ is a material parameter and $\Omega_{ik} = \partial_i(v_{sk}) - \partial_k(v_{si}) = \epsilon_{ikj}(\text{curl}\mathbf{v}_s)_j$.

It is easy to see that this dissipation potential verifies the requirements (3), (4) and that $\frac{\partial \Xi^{(HV)}}{\partial H_{w_i}}$ with an appropriately chosen scalar material parameter $\Lambda^{(HVBK)}(x^{(HVBK)})$ becomes indeed the dissipative part of the familiar Hall-Vinen mutual friction force $-\frac{B\rho_n}{\rho|\boldsymbol{\omega}} \boldsymbol{\omega} \times (\boldsymbol{\omega} \times E_{v_s})$, where B is the dissipative mutual friction coefficient.

4.4 GENERIC Time Evolution Equations

By writing down explicitly the nondissipative part of (1) with the state variables (36) and their Hamiltonian kinematics (39) we obtain the nondissipative part of (23) (except that the

fields \mathbf{u} and \mathbf{v}_s are now denoted \mathbf{m} and \mathbf{m}_s) and

$$\begin{aligned} \frac{\partial n}{\partial t} &= \partial_k (E_{w_k}) \\ \frac{\partial w_i}{\partial t} &= \partial_i (E_n) - \frac{1}{n} (\mathbf{E}_w \times \text{curl} \mathbf{w})_i \end{aligned} \tag{43}$$

governing the nondissipative time evolution of the new fields n and \mathbf{w} . By using (38), the time evolution of the overall momentum field \mathbf{u} is governed by the second equation in (23) with \mathbf{T} replaced by

$$T_{ik}^{(HVBK)} = m_i E_{m_k} + p^{(HVBK)} \delta_{ik} + \tau_{ik}^{(ndHVBK)} + \tau_{ik}^{(d)} \tag{44}$$

$$p^{(HVBK)} = -e + \rho E_\rho + \eta E_\eta + m_k E_{m_k} + n E_n \tag{45}$$

$$\tau_{ik}^{(ndHVBK)} = v_{si} E_{v_{sk}} - w_i E_{w_k} + v_{si} (\text{curl} E_\omega)_k - \partial_i (v_{sl}) \epsilon_{mlk} E_{\omega_m} \tag{46}$$

where $\boldsymbol{\omega} = \text{curl} \mathbf{v}_s = -\text{curl} \mathbf{w}$, and

$$\sigma = X_k^{(e)} \Xi_{X_k^{(e)}} + X_{kj}^{(u)} \Xi_{X_{kj}^{(u)}} + X_k^{(v_s)} \Xi_{X_k^{(v_s)}} + X_k^{(w)} \Xi_{X_k^{(w)}} \tag{47}$$

The full GENERIC equation governing the time evolution of $\mathbf{w}(\mathbf{r})$ is

$$\frac{\partial w_i}{\partial t} = \partial_k (E_n) - \frac{1}{n} (\mathbf{E}_w \times \text{curl} \mathbf{w})_i + \frac{\partial \Xi^{(HV)}}{\partial H_{w_i}} \tag{48}$$

4.5 Conjugate State Variables

In order that the time evolution equations that have arisen in Sect. 4.4 above are the same as the HVBK equations the fundamental thermodynamic relation has to be appropriately chosen. In particular, the thermodynamic potential has to depend also on \mathbf{w} , on $\boldsymbol{\omega} = -\text{curl} \mathbf{w}$, and it has to imply

$$\begin{aligned} E_n &= 0 \\ E_{w_j} &= -\rho E_{u_j} + E_{v_{sj}} = -(\rho_n v_{nj} + \rho_s v_{sj} + \text{curl} (E_{\text{curl} \mathbf{w}})_j) \end{aligned} \tag{49}$$

guaranteeing that the relation $n = \rho$ holds for all $t > 0$ provided it holds for $t = 0$. From (48) (with $E_n = 0$) we also see that $-\frac{1}{n} E_w$ has the physical interpretation of the vortex line velocity.

5 Two-Point Extension of the Landau-Tisza Model

Our intention in this section is to develop a general framework for a model of quantum turbulence in which the fluid small-scale microstructure is characterized in more details than in the HVBK model. In particular, we want to include explicitly the transport in both position space and scales. The question of a direct observability of the microstructural details involved in the model will not be addressed in this paper. We only show how the microstructure influences the overall flow of both mass and energy. Observations of the overall flow thus provide an indirect experimental access to the microstructural details.

Recasting the HVBK equations into the GENERIC form, made in Sect. 4 above, provides an additional support for their physical correctness. In particular, the recasting constitutes a proof of the agreement of one specific prediction of the model with one specific experimental observation (the observation that the fluid approaches, as $t \rightarrow \infty$, equilibrium states at which their behavior is found to be well described by equilibrium thermodynamics). The GENERIC framework has not been however involved in the initial physical insight that led to the HVBK model (see [26, 27]). In this section we shall develop an alternative nonlocal extension of the Landau-Tisza two-fluid model in which the GENERIC framework will serve as a guide. The Hamiltonian kinematics is put into the leading role. Even the choice of the extra fields characterizing the microstructure will come out first in the consideration of kinematics. The main result of this paper is a general framework for a new nonlocal extension of the Landau-Tisza model. Discussion of the specific physics needed to fill up the framework, i.e. to go through the third stage in GENERIC modeling in which the state variables are related to their conjugates, is only tentatively addressed in this paper.

5.1 State Variables x

As a point of departure we take a nonlocal view of the fluid obtained by replacing the fields (6) constituting state variables in the Landau-Tisza two-fluid model with two-point fields

$$x^{(2)} = (\varrho(\mathbf{x}_1, \mathbf{x}_2), \mathbf{v}_1(\mathbf{x}_1, \mathbf{x}_2), \mathbf{v}_2(\mathbf{x}_1, \mathbf{x}_2), \zeta(\mathbf{x}_1, \mathbf{x}_2), \boldsymbol{\chi}_1(\mathbf{x}_1, \mathbf{x}_2), \boldsymbol{\chi}_2(\mathbf{x}_1, \mathbf{x}_2)) \tag{50}$$

$\mathbf{x}_1 \in \mathbb{R}^3, \mathbf{x}_2 \in \mathbb{R}^3$. By ϱ we denote the two-point mass density, $(\mathbf{v}_1, \mathbf{v}_2)$ denotes the two-point overall momentum, ζ the two-point eta-function and $(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2)$ the two-point superfluid velocity. The following relations are required to hold:

$$\begin{aligned} \varrho(\mathbf{x}_1, \mathbf{x}_2) &= \varrho(\mathbf{x}_2, \mathbf{x}_1) \\ \zeta(\mathbf{x}_1, \mathbf{x}_2) &= \zeta(\mathbf{x}_2, \mathbf{x}_1) \\ \mathbf{v}_1(\mathbf{x}_2, \mathbf{x}_1) &= \mathbf{v}_2(\mathbf{x}_1, \mathbf{x}_2) \\ \boldsymbol{\chi}_1(\mathbf{x}_2, \mathbf{x}_1) &= \boldsymbol{\chi}_2(\mathbf{x}_1, \mathbf{x}_2) \end{aligned} \tag{51}$$

The parity transformation (15) becomes

$$\mathbf{I} : (\varrho, \mathbf{v}_1, \mathbf{v}_2, \zeta, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2) \mapsto (\varrho, -\mathbf{v}_1, -\mathbf{v}_2, \zeta, -\boldsymbol{\chi}_1, -\boldsymbol{\chi}_2) \tag{52}$$

The state variables (50) express more details than the state variables (6). In other words, the level of description on which (50) play the role of state variables is more microscopic than the one on which (6) play the same role.

In order to arrive at the set of state variables consisting of the overall (large-scale) variables (we shall denote them by the symbol $\hat{x}^{(1)}$) and microstructural (small-scale) variables (denoted by the symbol $\hat{x}^{(2)}$), we make the following two steps:

Step 1. We use both (6) and (50) as state variables. This means that the state variables are:

$$x = (x^{(1)}, x^{(2)}) \tag{53}$$

Both descriptions remain completely independent of each other.

Step 2. We now combine appropriately $x^{(1)}$ and $x^{(2)}$ and arrive at

$$\widehat{x} = (\widehat{x}^{(1)}, \widehat{x}^{(2)}) \tag{54}$$

where the symbol $\widehat{x}^{(1)}$ represents overall (large-scale) state variables and $\widehat{x}^{(2)}$ represents microstructural (small-scale) state variables. The passage $x \rightarrow \widehat{x}$ will be made as a one-to-one transformation.

In order to find the mapping $x \rightarrow \widehat{x}$, we ask first the following question: How do the two-point fields (50) reduce to the one-point fields (6)? We shall use for this purpose the following projection:

$$\begin{aligned} \rho(\mathbf{r}) &= \int d\mathbf{x}_1 \int d\mathbf{x}_2 [\varrho \delta(\mathbf{r} - \mathbf{x}_1) + \varrho \delta(\mathbf{r} - \mathbf{x}_2)] \\ \eta(\mathbf{r}) &= \int d\mathbf{x}_1 \int d\mathbf{x}_2 [\zeta \delta(\mathbf{r} - \mathbf{x}_1) + \zeta \delta(\mathbf{r} - \mathbf{x}_2)] \\ \mathbf{u}(\mathbf{r}) &= \int d\mathbf{x}_1 \int d\mathbf{x}_2 [\mathbf{v}_1 \delta(\mathbf{r} - \mathbf{x}_1) + \mathbf{v}_2 \delta(\mathbf{r} - \mathbf{x}_2)] \\ \mathbf{v}_s(\mathbf{r}) &= \frac{1}{2} \int d\mathbf{x}_1 \int d\mathbf{x}_2 [\chi_1 \delta(\mathbf{r} - \mathbf{x}_1) + \chi_2 \delta(\mathbf{r} - \mathbf{x}_2)] \end{aligned} \tag{55}$$

The appearance of 1/2 in the last equation is due to the fact that the field \mathbf{v}_s is the intensive variable and the remaining hydrodynamic fields are extensive variables.

The overall (large-scale) variables $\widehat{x}^{(1)}$ and the microstructural (small-scale) variables $\widehat{x}^{(2)}$ are now introduced by

$$\begin{aligned} \widehat{\rho}(\mathbf{r}) &= \rho(\mathbf{r}) + \int d\mathbf{x}_1 \int d\mathbf{x}_2 [\varrho \delta(\mathbf{r} - \mathbf{x}_1) + \varrho \delta(\mathbf{r} - \mathbf{x}_2)] \\ \widehat{\eta}(\mathbf{r}) &= \eta(\mathbf{r}) + \int d\mathbf{x}_1 \int d\mathbf{x}_2 [\zeta \delta(\mathbf{r} - \mathbf{x}_1) + \zeta \delta(\mathbf{r} - \mathbf{x}_2)] \\ \widehat{\mathbf{u}}(\mathbf{r}) &= \mathbf{u}(\mathbf{r}) + \int d\mathbf{x}_1 \int d\mathbf{x}_2 [\mathbf{v}_1 \delta(\mathbf{r} - \mathbf{x}_1) + \mathbf{v}_2 \delta(\mathbf{r} - \mathbf{x}_2)] \\ \widehat{\mathbf{v}}_s(\mathbf{r}) &= \frac{1}{3} \left[\mathbf{v}_s(\mathbf{r}) + \int d\mathbf{x}_1 \int d\mathbf{x}_2 (\chi_1 \delta(\mathbf{r} - \mathbf{x}_1) + \chi_2 \delta(\mathbf{r} - \mathbf{x}_2)) \right] \end{aligned} \tag{56}$$

$$\begin{aligned} \widehat{\varrho}(\mathbf{x}_1, \mathbf{x}_2) &= \varrho(\mathbf{x}_1, \mathbf{x}_2) \\ \widehat{\zeta}(\mathbf{x}_1, \mathbf{x}_2) &= \zeta(\mathbf{x}_1, \mathbf{x}_2) \\ \widehat{\mathbf{v}}_1(\mathbf{x}_1, \mathbf{x}_2) &= \mathbf{v}_1(\mathbf{x}_1, \mathbf{x}_2) \\ \widehat{\mathbf{v}}_2(\mathbf{x}_1, \mathbf{x}_2) &= \mathbf{v}_2(\mathbf{x}_1, \mathbf{x}_2) \\ \widehat{\chi}_1(\mathbf{x}_1, \mathbf{x}_2) &= \chi_1(\mathbf{x}_1, \mathbf{x}_2) \\ \widehat{\chi}_2(\mathbf{x}_1, \mathbf{x}_2) &= \chi_2(\mathbf{x}_1, \mathbf{x}_2) \end{aligned}$$

The multiplicative factor 1/3 appears in the fourth equation in (56) again due to the fact that \mathbf{v}_s is the intensive state variable while all the remaining state variables are extensive.

We note that the transformation $x \rightarrow \widehat{x}$ is indeed one-to-one.

Next, we make two another one-to-one reformulations of (54). First, we replace the coordinates $(\mathbf{x}_1, \mathbf{x}_2)$ with (\mathbf{x}, \mathbf{R}) by using the transformation

$$\mathbf{x} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2); \quad \mathbf{R} = \mathbf{x}_2 - \mathbf{x}_1 \tag{57}$$

Second, we replace $(\widehat{v}_1, \widehat{v}_2, \widehat{\chi}_1, \widehat{\chi}_2)$ by $(\widehat{U}, \widehat{W}, \widehat{U}_s, \widehat{W}_s)$:

$$\begin{aligned} \widehat{U} &= \widehat{v}_1 + \widehat{v}_2 \\ \widehat{W} &= \frac{1}{2}(\widehat{v}_2 - \widehat{v}_1) \\ \widehat{U}_s &= \widehat{\chi}_1 + \widehat{\chi}_2 \\ \widehat{W}_s &= \frac{1}{2}(\widehat{\chi}_2 - \widehat{\chi}_1) \end{aligned} \tag{58}$$

The state variables replacing (53) are thus

$$\widehat{x} = (\widehat{\rho}(\mathbf{r}), \widehat{\mathbf{u}}(\mathbf{r}), \widehat{\eta}(\mathbf{r}), \widehat{v}_s(\mathbf{r}), \widehat{q}(\mathbf{r}, \mathbf{R}), \widehat{\zeta}(\mathbf{r}, \mathbf{R}), \widehat{U}(\mathbf{r}, \mathbf{R}), \widehat{W}(\mathbf{r}, \mathbf{R}), \widehat{U}_s(\mathbf{r}, \mathbf{R}), \widehat{W}_s(\mathbf{r}, \mathbf{R})) \tag{59}$$

From the physical point of view, the fields $(\widehat{\rho}(\mathbf{r}), \widehat{\mathbf{u}}(\mathbf{r}), \widehat{\eta}(\mathbf{r}), \widehat{v}_s(\mathbf{r}))$ are the fields used as state variables in the classical Landau-Tisza theory. The potentials $N(\widehat{x})$, $H(\widehat{x})$, and $E(\widehat{x})$ are given by

$$N(\widehat{x}) = \frac{1}{M_{mol}} \int d\mathbf{r} \widehat{\rho}(\mathbf{r}) \tag{60}$$

where M_{mol} is the molecular weight and

$$H(\widehat{x}) = \int d\mathbf{r} \widehat{\eta}(\mathbf{r}) \tag{61}$$

The third function

$$E(\widehat{x}) = \int d\mathbf{r} e(\widehat{x}; \mathbf{r}) \tag{62}$$

The remaining fields $(\widehat{q}(\mathbf{r}, \mathbf{R}), \widehat{\zeta}(\mathbf{r}, \mathbf{R}), \widehat{U}(\mathbf{r}, \mathbf{R}), \widehat{W}(\mathbf{r}, \mathbf{R}), \widehat{U}_s(\mathbf{r}, \mathbf{R}), \widehat{W}_s(\mathbf{r}, \mathbf{R}))$ are extra state variables expressing an extra information about nonlocal aspects of the superfluid that is unseen in the classical theory. Roughly speaking, $\widehat{q}(\mathbf{r}, \mathbf{R})$ is related to mass fluctuations, $\widehat{\zeta}(\mathbf{r}, \mathbf{R})$ to the temperature fluctuations, $\widehat{U}(\mathbf{r}, \mathbf{R})$ (resp. $\widehat{U}_s(\mathbf{r}, \mathbf{R})$) to the overall (resp. superfluid) velocity fluctuations, and $\widehat{W}(\mathbf{r}, \mathbf{R})$ (resp. $\widehat{W}_s(\mathbf{r}, \mathbf{R})$) to the overall fluctuations of the gradient of the overall (resp. superfluid) velocity. We shall say more about the physical meaning of (59) in Sect. 5.5. after establishing the general framework of the equations governing their time evolution.

In this paper we shall limit ourselves only to the nonlocal behavior of the superfluid component of the liquid helium and thus the state variables used below in this section are:

$$\widehat{x} = (\widehat{\rho}(\mathbf{r}), \widehat{\mathbf{u}}(\mathbf{r}), \widehat{\eta}(\mathbf{r}), \widehat{v}_s(\mathbf{r}), \widehat{q}(\mathbf{r}, \mathbf{R}), \widehat{\zeta}(\mathbf{r}, \mathbf{R}), \widehat{U}_s(\mathbf{r}, \mathbf{R}), \widehat{W}_s(\mathbf{r}, \mathbf{R})) \tag{63}$$

5.2 Kinematics L

In this section we find the Poisson bracket in which the kinematics of the state variables (63) is expressed. We proceed as follows: first, we find the Poisson bracket $\{A, B\}^{(2)}$ expressing the kinematics of (50). The Poisson bracket $\{A, B\}^{(1\&2)}$ corresponding to (53) is then

$$\{A, B\}^{(1\&2)} = \{A, B\}^{(1)} + \{A, B\}^{(2)} \tag{64}$$

where $\{A, B\}^{(1)}$ is given in (16). Since the passage from (53) to (59) is made by a sequence of one-to-one transformations, the Poisson bracket expressing the kinematics of (59) is derived from $\{A, B\}^{(1\&2)}$. By using the transformation (56) we arrive first from $\{A, B\}^{(1\&2)}$ to $\{\widehat{A}, \widehat{B}\}$ and then by using the transformations (57), (58), and restriction of the dependence of the functions A, B , we arrive from $\{\widehat{A}, \widehat{B}\}$ to $\{A, B\}$ expressing kinematics of the state variables (63).

The only unknown in the two steps described above is the bracket $\{A, B\}^{(2)}$. But this bracket, as well as the bracket $\{A, B\}^{(1)}$, can be found in [18]. Indeed, the state variables used in [18] are fields (6) where $\mathbf{r} \in \mathbb{R}^N$. The Poisson bracket (16) corresponds to $N = 3$ and the Poisson bracket $\{A, B\}^{(2)}$ corresponds to $N = 6$. The bracket $\{A, B\}^{(2)}$ is thus $N = 6$ version of the bracket $\{A, B\}^{(1)}$ and is therefore given by

$$\begin{aligned} \{A, B\}^{(2)} = & \sum_{\gamma=1}^2 \sum_{\alpha=1, \alpha \neq \gamma}^2 \int d\mathbf{x}_1 \int d\mathbf{x}_2 [v_{\gamma i}(\partial_{\gamma j}(A_{v_{\gamma i}})B_{v_{\gamma j}} - \partial_{\gamma j}(B_{v_{\gamma i}})A_{v_{\gamma j}}) \\ & + v_{\gamma i}(\partial_{\alpha j}(A_{v_{\gamma i}})B_{v_{\alpha j}} - \partial_{\alpha j}(B_{v_{\gamma i}})A_{v_{\alpha j}}) + \varrho(\partial_{\gamma j}(A_{\varrho})B_{v_{\gamma j}} - \partial_{\gamma j}(B_{\varrho})A_{v_{\gamma j}}) \\ & + \zeta(\partial_{\gamma j}(A_{\zeta})B_{v_{\gamma j}} - \partial_{\gamma j}(B_{\zeta})A_{v_{\gamma j}}) + \chi_{\gamma i}(\partial_{\gamma j}(A_{\chi_{\gamma i}})B_{\chi_{\gamma j}} - \partial_{\gamma j}(B_{\chi_{\gamma i}})A_{\chi_{\gamma j}}) \\ & + \chi_{\gamma i}(\partial_{\alpha j}(A_{v_{\gamma i}})B_{\chi_{\alpha j}} - \partial_{\alpha j}(B_{v_{\gamma i}})A_{\chi_{\alpha j}}) + \partial_{\gamma i}(\chi_{\gamma j})(A_{v_{\gamma i}}B_{\chi_{\gamma j}} - B_{v_{\gamma i}}A_{\chi_{\gamma j}}) \\ & + \partial_{\gamma i}(\chi_{\alpha j})(A_{v_{\gamma i}}B_{\chi_{\alpha j}} - B_{v_{\gamma i}}A_{\chi_{\alpha j}}) - (A_{\varrho}\partial_{\gamma i}(B_{\chi_{\gamma i}}) - B_{\varrho}\partial_{\gamma i}(A_{\chi_{\gamma i}}))] \end{aligned} \tag{65}$$

We now make the two steps described above. First, we use the one-to-one transformation (56) to arrive from the Poisson bracket $\{A, B\}^{(1\&2)}$ at the Poisson bracket

$$\{\widehat{A}, \widehat{B}\} = \{\widehat{A}, \widehat{B}\}^{(1)} + \{\widehat{A}, \widehat{B}\}^{(12)} \tag{66}$$

where $\{\widehat{A}, \widehat{B}\}^{(1)}$ is the Poisson bracket (16) in which the fields $(\rho(\mathbf{r}), \mathbf{u}(\mathbf{r}), \eta(\mathbf{r}), \mathbf{v}_s(\mathbf{r}))$ are replaced by $\widehat{\rho}(\mathbf{r}), \widehat{\mathbf{u}}(\mathbf{r}), \widehat{\eta}(\mathbf{r}), \widehat{\mathbf{v}}_s(\mathbf{r})$ and

$$\begin{aligned} \{\widehat{A}, \widehat{B}\}^{(12)} = & \int d\mathbf{x}_1 \int d\mathbf{x}_2 \widehat{\varrho} [\partial_{1j}(A_{\widehat{\varrho}})B_{\widehat{u}_j(x_1)} - \partial_{1j}(B_{\widehat{\varrho}})A_{\widehat{u}_j(x_1)}] \\ & + \widehat{\varrho} [\partial_{2j}(A_{\widehat{\varrho}})B_{\widehat{u}_j(x_2)} - \partial_{2j}(B_{\widehat{\varrho}})A_{\widehat{u}_j(x_2)}] + \widehat{\zeta} [\partial_{1j}(A_{\widehat{\zeta}})B_{\widehat{u}_j(x_1)} - \partial_{1j}(B_{\widehat{\zeta}})A_{\widehat{u}_j(x_1)}] \\ & + \widehat{\zeta} [\partial_{2j}(A_{\widehat{\zeta}})B_{\widehat{u}_j(x_2)} - \partial_{2j}(B_{\widehat{\zeta}})A_{\widehat{u}_j(x_2)}] \\ & + \widehat{\chi}_{1i} [\partial_{1j}(A_{\widehat{u}_i(x_1)})B_{\widehat{\chi}_{1j}} - \partial_{1j}(B_{\widehat{u}_i(x_1)})A_{\widehat{\chi}_{1j}}] \\ & + \widehat{\chi}_{2i} [\partial_{2j}(A_{\widehat{u}_i(x_2)})B_{\widehat{\chi}_{2j}} - \partial_{2j}(B_{\widehat{u}_i(x_2)})A_{\widehat{\chi}_{2j}}] \\ & + \partial_{1i}(\widehat{\chi}_{1j}) [A_{\widehat{u}_i(x_1)}B_{\widehat{\chi}_{1j}} - B_{\widehat{u}_i(x_1)}A_{\widehat{\chi}_{1j}}] \\ & + \partial_{2i}(\widehat{\chi}_{2j}) [A_{\widehat{u}_i(x_2)}B_{\widehat{\chi}_{2j}} - B_{\widehat{u}_i(x_2)}A_{\widehat{\chi}_{2j}}] + \partial_{1i}(\widehat{\chi}_{2j}) [A_{\widehat{u}_i(x_1)}B_{\widehat{\chi}_{2j}} - B_{\widehat{u}_i(x_1)}A_{\widehat{\chi}_{2j}}] \\ & + \partial_{2i}(\widehat{\chi}_{1j}) [A_{\widehat{u}_i(x_2)}B_{\widehat{\chi}_{1j}} - B_{\widehat{u}_i(x_2)}A_{\widehat{\chi}_{1j}}] \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{3}(\widehat{\chi}_{2j})[\partial_{1i}(A_{\widehat{u}_i(x_1)})B_{\widehat{v}_{sj}(x_2)} - \partial_{1i}(B_{\widehat{u}_i(x_1)})A_{\widehat{v}_{sj}(x_2)}] \\
 & -\frac{1}{3}(\widehat{\chi}_{1j})[\partial_{2i}(A_{\widehat{u}_i(x_2)})B_{\widehat{v}_{sj}(x_1)} - \partial_{2i}(B_{\widehat{u}_i(x_2)})A_{\widehat{v}_{sj}(x_1)}] \\
 & - [A_{\widehat{\rho}}\partial_{1i}(B_{\widehat{\chi}_{1i}}) - B_{\widehat{\rho}}\partial_{1i}(A_{\widehat{\chi}_{1i}})] - [A_{\widehat{\rho}}\partial_{2i}(B_{\widehat{\chi}_{2i}}) - B_{\widehat{\rho}}\partial_{2i}(A_{\widehat{\chi}_{2i}})] \\
 & -\frac{1}{3}[A_{\widehat{\rho}}\partial_{1i}(B_{\widehat{v}_{si}(x_1)}) - B_{\widehat{\rho}}\partial_{1i}(A_{\widehat{v}_{si}(x_1)})] - \frac{1}{3}[A_{\widehat{\rho}}\partial_{2i}(B_{\widehat{v}_{si}(x_2)}) - B_{\widehat{\rho}}\partial_{2i}(A_{\widehat{v}_{si}(x_2)})] \\
 & - [A_{\widehat{\rho}(x_1)}\partial_{1i}(B_{\widehat{\chi}_{1i}}) - B_{\widehat{\rho}(x_1)}\partial_{1i}(A_{\widehat{\chi}_{1i}})] - [A_{\widehat{\rho}(x_2)}\partial_{2i}(B_{\widehat{\chi}_{2i}}) - B_{\widehat{\rho}(x_2)}\partial_{2i}(A_{\widehat{\chi}_{2i}})] \\
 & - [A_{\widehat{\rho}(x_1)}\partial_{2i}(B_{\widehat{\chi}_{2i}}) - B_{\widehat{\rho}(x_1)}\partial_{2i}(A_{\widehat{\chi}_{2i}})] \\
 & - [A_{\widehat{\rho}(x_2)}\partial_{1i}(B_{\widehat{\chi}_{1i}}) - B_{\widehat{\rho}(x_2)}\partial_{1i}(A_{\widehat{\chi}_{1i}})] \tag{67}
 \end{aligned}$$

In the calculations leading to (66) we have used

$$\begin{aligned}
 A_{\rho(r)} &= A_{\widehat{\rho}(r)} \\
 A_{\eta(r)} &= A_{\widehat{\eta}(r)} \\
 A_{\mathbf{u}(r)} &= A_{\widehat{\mathbf{u}}(r)} \\
 A_{v_s(r)} &= \frac{1}{3}A_{\widehat{v}_s(r)} \\
 A_{\varrho(x_1,x_2)} &= A_{\widehat{\varrho}(x_1,x_2)} + A_{\widehat{\rho}(x_1)} + A_{\widehat{\rho}(x_2)} \\
 A_{\zeta(x_1,x_2)} &= A_{\widehat{\zeta}(x_1,x_2)} + A_{\widehat{\eta}(x_1)} + A_{\widehat{\eta}(x_2)} \\
 A_{\chi_1(x_1,x_2)} &= A_{\widehat{\chi}_1(x_1,x_2)} + \frac{1}{3}A_{\widehat{v}_s(x_1)} \\
 A_{\chi_2(x_1,x_2)} &= A_{\widehat{\chi}_2(x_1,x_2)} + \frac{1}{3}A_{\widehat{v}_s(x_2)}
 \end{aligned} \tag{68}$$

and the same relations for the derivatives of the function B . Since we shall restrict in the next step our interest to the nonlocal extension of only the superfluid component, we have already omitted in (67), for the sake of simplicity, the terms involving the derivatives with respect to \mathbf{v}_1 and \mathbf{v}_2 .

From now on all the state variables are with hats. In order to simplify the notation we shall omit hereafter the hats.

In the second step we use the one-to one transformations (57) and (58) and the restriction of the dependence of the functions A and B to arrive from the bracket $\widehat{\{A, B\}}$ given in (67) to the bracket $\{A, B\}$ expressing kinematics of the state variables (63). In the calculations we use

$$\frac{\partial}{\partial \mathbf{x}_1} = \frac{1}{2} \frac{\partial}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{R}} \tag{69}$$

$$\frac{\partial}{\partial \mathbf{x}_2} = \frac{1}{2} \frac{\partial}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{R}}$$

$$\begin{aligned}
 A_{\chi_1(x_1,x_2)} &= A_{U_s(x_1,x_2)} - \frac{1}{2}A_{W_s(x_1,x_2)} \\
 A_{\chi_2(x_1,x_2)} &= A_{U_s(x_1,x_2)} + \frac{1}{2}A_{W_s(x_1,x_2)}
 \end{aligned} \tag{70}$$

and

$$\begin{aligned}
 A_{\eta(x_1)} &= A_{\eta(x)} - \frac{1}{2} R_j \frac{\partial}{\partial x_j} A_{\eta(x)} \\
 A_{\eta(x_2)} &= A_{\eta(x)} + \frac{1}{2} R_j \frac{\partial}{\partial x_j} A_{\eta(x)}
 \end{aligned}
 \tag{71}$$

In (71) we are omitting the terms involving the second and higher order derivatives with respect to the position coordinate \mathbf{x} . The relations (71) are used for all the fields and all relations (69), (70) and (71) are used for the derivatives of both A and B . After somewhat tedious but completely straightforward calculations we arrive at

$$\{A, B\} = \{A, B\}^{(1)} + \{A, B\}^{(12)}
 \tag{72}$$

where $\{A, B\}^{(1)}$ is the Poisson bracket (16) and

$$\begin{aligned}
 \{A, B\}^{(12)} &= \int d\mathbf{x} \int d\mathbf{R} \left[\varrho [\partial_j (A_\varrho) B_{u_j} - \partial_j (B_\varrho) A_{u_j}] \right. \\
 &\quad + \varrho R_k [\partial_{R_j} (A_\varrho) \partial_k (B_{u_j}) - \partial_{R_j} (B_\varrho) \partial_k (A_{u_j})] \\
 &\quad + \zeta [\partial_j (A_\zeta) B_{u_j} - \partial_j (B_\zeta) A_{u_j}] + \zeta R_k [\partial_{R_j} (A_\zeta) \partial_k (B_{u_j}) - \partial_{R_j} (B_\zeta) \partial_k (A_{u_j})] \\
 &\quad + U_{si} [\partial_j (A_{u_i}) B_{U_{sj}} - \partial_j (B_{u_i}) A_{U_{sj}}] + W_{si} [\partial_j (A_{u_i}) B_{W_{sj}} - \partial_j (B_{u_i}) A_{W_{sj}}] \\
 &\quad + \frac{1}{8} U_{si} R_k [\partial_j \partial_k (A_{u_i}) B_{W_{sj}} - \partial_j \partial_k (B_{u_i}) A_{W_{sj}}] \\
 &\quad + \frac{1}{2} W_{si} R_k [\partial_j \partial_k (A_{u_i}) B_{U_{sj}} - \partial_j \partial_k (B_{u_i}) A_{U_{sj}}] \\
 &\quad + \partial_i (U_{sj}) [A_{u_i} B_{U_{sj}} - B_{u_i} A_{U_{sj}}] + \partial_i (W_{sj}) [A_{u_i} B_{W_{sj}} - B_{u_i} A_{W_{sj}}] \\
 &\quad + \partial_{R_i} (U_{sj}) R_k [\partial_k (A_{u_i}) B_{U_{sj}} - \partial_k (B_{u_i}) A_{U_{sj}}] \\
 &\quad + \partial_{R_i} (W_{sj}) R_k [\partial_k (A_{u_i}) B_{W_{sj}} - \partial_k (B_{u_i}) A_{W_{sj}}] \\
 &\quad - \frac{1}{3} U_{sj} [\partial_i (A_{u_i}) B_{v_{sj}} - \partial_i (B_{u_i}) A_{v_{sj}}] \\
 &\quad + \frac{1}{24} U_{sj} R_k R_l [\partial_i \partial_k (A_{u_i}) \partial_l (B_{v_{sj}}) - \partial_i \partial_k (B_{u_i}) \partial_l (A_{v_{sj}})] \\
 &\quad - \frac{1}{6} W_{sj} R_k [\partial_i (A_{u_i}) \partial_k (B_{v_{sj}}) - \partial_i (B_{u_i}) \partial_k (A_{v_{sj}})] \\
 &\quad + \frac{1}{6} W_{sj} R_k [\partial_i \partial_k (A_{u_i}) B_{v_{sj}} - \partial_i \partial_k (B_{u_i}) A_{v_{sj}}] - 2[A_\rho \partial_j (B_{U_{sj}}) - B_\rho \partial_j (A_{U_{sj}})] \\
 &\quad - 2[A_\rho \partial_{R_j} (B_{W_{sj}}) - B_\rho \partial_{R_j} (A_{W_{sj}})] - [A_\varrho \partial_j (B_{U_{sj}}) - B_\varrho \partial_j (A_{U_{sj}})] \\
 &\quad \left. - [A_\varrho \partial_{R_j} (B_{W_{sj}}) - B_\varrho \partial_{R_j} (A_{W_{sj}})] - \frac{2}{3} [A_\varrho \partial_j (B_{v_{sj}}) - B_\varrho \partial_j (A_{v_{sj}})] \right]
 \end{aligned}
 \tag{73}$$

We have thus arrived at the bracket expressing kinematics of the state variables (63). We are certain that the bracket (72) is a Poisson bracket. This is because the bracket (72) has been obtained from the Poisson bracket $\{A, B\}^{(1\&2)}$ given in (64) (which clearly is a Poisson bracket since both of the brackets on the right hand side of (64) are separately

Poisson brackets—see [18]) by using a one-to-one transformation and a restriction put on the dependence of A and B \widehat{x} that is preserved in the bracket. We note that a direct verification of the skew-symmetry of the bracket (72) is straightforward but a direct verification of the Jacobi identity would require very long and tedious calculations.

5.3 Dissipation Potential Ξ

Without entering details of the physics involved in the dissipative processes taking place in ^4He superfluid subjected to a turbulent flow (see for example the physics that leads in [28, 29], in the context of the HVBK model, to the Hall-Vinen dissipative mutual friction force (42)), only the requirements (3), (4) appearing in the abstract formulation of the GENERIC framework provide a basis for its discussion. On this basis we suggest below an admissible dissipation potential.

As in Sect. 3.3, we begin with the identification of thermodynamic forces. In addition to $X^{(\varrho)}$, $X^{(u)}$, and $X^{(v_s)}$ used in (18) in the classical Landau theory, we introduce

$$\begin{aligned}
 X^{(\varrho)} &= H_{\varrho} \\
 X^{(\zeta)} &= H_{\zeta} \\
 X_i^{(U_s)} &= H_{U_{si}} \\
 X_i^{(W_s)} &= H_{W_{si}} \\
 \mathfrak{X}_i^{(\varrho)} &= \partial_{R_i} H_{\varrho} \\
 \mathfrak{X}_i^{(\zeta)} &= \partial_{R_i} H_{\zeta} \\
 \mathfrak{X}_{ij}^{(U_s)} &= \partial_{R_j} H_{U_{si}} \\
 \mathfrak{X}_{ij}^{(W_s)} &= \partial_{R_j} H_{W_{si}}
 \end{aligned}
 \tag{74}$$

Since $(\varrho, \zeta, U_s, W_s)$ represent a small-scale structure that is present only in the turbulent fluid, we expect that all these forces will disappear at equilibrium at which the thermodynamic potential (5) reaches its minimum and all the thermodynamic forces equal zero. Indeed, this will be the case if the fundamental thermodynamic relation is chosen in such a way that $H_x = 0$ implies $x = 0$ for $x = \varrho, \zeta, U_s, W_s$.

Next, we introduce a dissipation potential $\Xi^{(2)}$ as a sum of the dissipation potential (19) and a real valued function of the thermodynamic forces (74) satisfying the properties (3) and the degeneracy requirement needed to guarantee (4). We shall leave it unspecified.

5.4 The GENERIC Time Evolution Equations

We have collected now everything that we need to write down the time evolution equations of the extended model. The equations are written on the same level of abstractness as (23) are written for the standard Landau model. In both (23) and (77) below, the fundamental thermodynamic relation is left undetermined. For the sake of simplicity, we are omitting in the equations appearing below terms proportional to $\partial_i E_{ui}$ (i.e. we assume that the fluid is incompressible; $\partial_i E_{ui} = 0$). We shall comment about the effect of compressibility in Sect. 5.5. The energy E (whose specification as a function of the state variables (63) constitutes the fundamental thermodynamic relation) is written as

$$E = \int d\mathbf{r} \int d\mathbf{R} e(\rho, \mathbf{u}, \eta, \mathbf{v}_s, \varrho, \zeta, U_s, W_s; \mathbf{r}, \mathbf{R})
 \tag{75}$$

By the symbols e we denote the energy density that depends on both the position coordinate \mathbf{r} and the scale label coordinate \mathbf{R} . Similarly in the entropy representation we write

$$H = \int d\mathbf{r} \int d\mathbf{R} \eta(\rho, \mathbf{u}, e, \mathbf{v}_s, \varrho, \zeta, \mathbf{U}_s, \mathbf{W}_s; \mathbf{r}, \mathbf{R}) \tag{76}$$

where η denotes the eta-function density depending on both the position coordinate \mathbf{r} and the scale label coordinate \mathbf{R} .

By inserting (72) and the dissipation potential $\Xi^{(2)}$ introduced in Sect. 4.3 into (1) we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\partial_k J_k^{(2\rho)} \\ \frac{\partial \eta}{\partial t} &= -\partial_k (\eta E_{u_k}) - \partial_k J_k^{(2\eta r)} - \partial_{R_k} J_k^{(2\eta R)} + \sigma \\ \frac{\partial u_i}{\partial t} &= -\partial_k T_{ik}^{(2)} \\ \frac{\partial v_{si}}{\partial t} &= \left(\frac{\partial v_{si}}{\partial t}\right)^{(1)} - \frac{2}{3} \partial_i \int d\mathbf{R} E_\varrho \\ \frac{\partial \varrho}{\partial t} &= -\partial_j (\varrho E_{u_j}) - \frac{2}{3} \partial_j E_{v_{sj}} - \partial_j E_{U_{sj}} - \partial_{R_j} (\varrho R_k \partial_k E_{u_j}) \\ &\quad - \partial_{R_j} E_{W_{sj}} + \frac{\partial \Xi}{\partial H_\varrho} - \partial_{R_k} \left(\frac{\partial \Xi}{\partial (\partial_{R_k} H_\varrho)}\right) \\ \frac{\partial \zeta}{\partial t} &= -\partial_j (\zeta E_{u_j}) - \partial_{R_j} (\zeta R_k \partial_k E_{u_j}) + \frac{\partial \Xi}{\partial H_\zeta} - \partial_{R_k} \left(\frac{\partial \Xi}{\partial (\partial_{R_k} H_\zeta)}\right) \\ \frac{\partial U_{si}}{\partial t} &= -2\partial_i E_\rho - \partial_i E_\varrho - U_{sj} \partial_i E_{u_j} - \partial_j (U_{si}) E_{u_j} - \partial_{R_j} (U_{si}) R_k \partial_k E_{u_j} \\ &\quad - \frac{1}{2} W_{sj} R_k \partial_i \partial_k E_{u_j} + \frac{\partial \Xi}{\partial H_{U_{si}}} - \partial_{R_k} \left(\frac{\partial \Xi}{\partial (\partial_{R_k} H_{U_{si}})}\right) \\ \frac{\partial W_{si}}{\partial t} &= -\partial_{R_i} E_\varrho - W_{sj} \partial_i E_{u_j} - \partial_j (W_{si}) E_{u_j} - \partial_{R_j} (W_{si}) R_k \partial_k E_{u_j} - \frac{1}{8} U_{sj} R_k \partial_i \partial_k E_{u_j} \\ &\quad + \frac{\partial \Xi}{\partial H_{W_{si}}} - \partial_{R_k} \left(\frac{\partial \Xi}{\partial (\partial_{R_k} H_{W_{si}})}\right) \end{aligned} \tag{77}$$

and

$$\frac{\partial e}{\partial t} = -\partial_k J_k^{(2e)} - \partial_{R_j} \Upsilon_k^{(2e)} \tag{78}$$

By $\left(\frac{\partial v_{si}}{\partial t}\right)^{(1)}$ we denote the right hand side of the fourth equation in (23). The mass flux $\mathbf{J}^{(2\rho)}$, the stress tensor $\mathbf{T}^{(2)}$, and the energy fluxes $\mathbf{J}^{(2e)}$, $\mathbf{\Upsilon}^{(2e)}$ are given by

$$J_k^{(2\rho)} = \rho E_{u_k} + E_{v_{sk}} + 2 \int d\mathbf{R} E_{U_{sk}} \tag{79}$$

$$T_{ik}^{(2)} = u_i E_{u_k} + p^{(2)} \delta_{ik} + \tau_{ik}^{(2nd)} + \tau_{ik}^{(2d)} \tag{80}$$

$$J_k^{(2e)} = (p^{(2)} + e) E_{u_k} + \tau_{kj}^{(2nd)} E_{u_j} + q_k^{(2nd)} + \tau_{kj}^{(d)} E_{u_j} + q_k^{(d)} \tag{81}$$

$$J_k^{(2\eta^r)} = H_e \Xi_{X_k^{(e)}}^{(2)} \tag{82}$$

$$J_k^{(2\eta^R)} = H_\varrho \Xi_{\partial_{R_k} X^{(\varrho)}}^{(2)} + H_\zeta \Xi_{\partial_{R_k} X^{(\zeta)}}^{(2)} + H_{U_{si}} \Xi_{\partial_{R_k} X_i^{(U_s)}}^{(2)} + H_{W_{si}} \Xi_{\partial_{R_k} X_i^{(W_s)}}^{(2)} \tag{83}$$

$$\Upsilon_j^{(2e)} = (2E_\rho + E_\varrho) E_{W_j} + (\varrho R_k E_\varrho + \zeta R_k E_\zeta) \partial_k E_{u_j} \tag{84}$$

where

$$p^{(2)} = -e + \rho E_\rho + \eta E_\eta + u_k E_{u_k} + \int d\mathbf{R} (\varrho E_\varrho + \zeta E_\zeta) + \tau^{(2nd)} \tag{85}$$

$$\begin{aligned} \tau^{(2nd)} = & - \int d\mathbf{R} \left[\frac{1}{3} U_{sl} E_{v_{sl}} + \frac{1}{6} W_{sj} R_m \partial_m E_{v_{sj}} \right. \\ & \left. + \frac{1}{24} \partial_m (U_{sj} R_m R_l \partial_l E_{v_{sj}}) + \frac{1}{6} \partial_m (W_{sj} R_m E_{v_{sj}}) \right] \end{aligned} \tag{86}$$

$$\begin{aligned} \tau_{ij}^{(2nd)} = & v_{si} E_{v_{sj}} + \int d\mathbf{R} \left[-\varrho R_j \partial_{R_i} E_\varrho - \zeta R_j \partial_{R_i} E_\zeta + U_{si} E_{U_{sj}} + W_{si} E_{W_{sj}} \right. \\ & + \partial_{R_i} (U_{sk}) R_j E_{U_{sk}} + \partial_{R_i} (W_{sk}) R_j E_{W_{sk}} - \frac{1}{8} \partial_k (U_{si} R_k E_{W_{sj}}) \\ & \left. - \frac{1}{2} \partial_k (W_{si} R_k E_{U_{sj}}) \right] \end{aligned} \tag{87}$$

$$\tau_{ij}^{(d)} = \frac{1}{H_e} \Lambda^{(u)} X_{ij}^{(u)} \tag{88}$$

$$q_i^{(d)} = \Lambda^{(e)} X_i^{(e)}$$

$$q_i^{(2nd)} = E_\rho E_{v_{si}} + \int d\mathbf{R} (E_\varrho E_{v_{si}} + 2E_\rho E_{U_{si}} + E_\varrho E_{U_{si}}) \tag{89}$$

and the entropy production

$$\begin{aligned} \sigma = & X_k^{(e)} \Xi_{X_k^{(e)}}^{(2)} + X_{kj}^{(u)} \Xi_{X_{kj}^{(u)}}^{(2)} + X_k^{(v_s)} \Xi_{X_k^{(v_s)}}^{(2)} \\ & + X^{(\varrho)} \Xi_{X^{(\varrho)}}^{(2)} + X^{(\zeta)} \Xi_{X^{(\zeta)}}^{(2)} + X_k^{(U_s)} \Xi_{X_k^{(U_s)}}^{(2)} + X_k^{(W_s)} \Xi_{X_k^{(W_s)}}^{(2)} \\ & + \mathfrak{X}_k^{(\varrho)} \Xi_{\mathfrak{X}_k^{(\varrho)}}^{(2)} + \mathfrak{X}_k^{(\zeta)} \Xi_{\mathfrak{X}_k^{(\zeta)}}^{(2)} + \mathfrak{X}_{kj}^{(U_s)} \Xi_{\mathfrak{X}_{kj}^{(U_s)}}^{(2)} + \mathfrak{X}_{kj}^{(W_s)} \Xi_{\mathfrak{X}_{kj}^{(W_s)}}^{(2)} \end{aligned} \tag{90}$$

is positive provided the dissipation potential $\Xi^{(2)}$ is a real valued function of the thermodynamic forces (18) and (74) satisfying the properties (3).

Equations (77)–(90) are the two-point extension of (23)–(32). We make three observations:

(i) In the absence of the microstructure (i.e. in the absence of the fields $\varrho(\mathbf{r}, \mathbf{R})$, $\zeta(\mathbf{r}, \mathbf{R})$, $U_s(\mathbf{r}, \mathbf{R})$, $W_s(\mathbf{r}, \mathbf{R})$) (77)–(90) reduce indeed to (23)–(32).

(ii) As $t \rightarrow \infty$, solutions to (77)–(90) approach equilibrium states at which the behavior of the ⁴He superfluid is well described by equilibrium thermodynamics. This is because (77)–(90) represent a particular realization of the GENERIC equation (1).

(iii) The stress tensor and the heat flux determining the overall fluid behavior as well as the energy and entropy fluxes in both the position space and the scales appear in (77)–(90) as explicit functions of the microstructural state variables and of their conjugates.

5.5 Conjugate State Variables

Now we turn to the thermodynamic part of the modeling providing the relation between the state variables and their conjugates. We shall limit ourselves only to a discussion of the physical interpretation of the microstructural (small-scale) state variables as it arises from the way they are defined in terms of the two-point hydrodynamic fields (see Sect. 5.1) and from their role, seen in (77)–(90), that they play in dynamics.

The field $\varrho(\mathbf{r}, \mathbf{R})$ represents a small-scale nonlocal variations of the overall mass density. From (77) we see that its conjugate is the only small-scale field that influences the time evolution of the superfluid velocity \mathbf{v}_s . All other new terms (new with respect to (23)) that appear on the right hand side of the fourth equation in (77) are terms proportional to $\partial_j E_{u_j}$. Due to our limitation to the incompressible fluids, these terms are omitted in (77). The field ϱ and its conjugate contribute also to the “vortex line tension” (i.e. the part of the stress tensor \mathbf{T} that appears as new in (80) with respect to (26)), to the heat flux (89), and to the inter-scale energy flux Υ (see (84)).

The conjugate of the field $\zeta(\mathbf{r}, \mathbf{R})$ represents the small-scale structure of the turbulence temperature, the field ζ itself expresses the entropy of the small-scale order created in the turbulent flow. The concepts of both the turbulence entropy and the turbulence temperature were discussed recently in [31, 32]. We note that the field ζ and its conjugate enter also the vortex line tension.

The field $U_s(\mathbf{r}, \mathbf{R})$ represents the small-scale structure of the superfluid velocity \mathbf{v}_s . Its conjugate plays the role of the velocity in the additional advection (occurring only in the turbulent flow) of the mass (see (79)) and all other fields. In the form $U_s E_{U_s}$ it also appears in the vortex line tension.

The field $W_s(\mathbf{r}, \mathbf{R})$ characterizes the small-scale structure of the gradient of the superfluid velocity \mathbf{v}_s . Indeed, the conjugate of $\rho_s \int d\mathbf{R} \mathbf{R} W_s$ can be seen as a superfluid velocity gradient. Consequently, the vector $\boldsymbol{\omega}$ defined by

$$\omega_k = \epsilon_{ijk} \Omega_{ij} \quad (91)$$

where the tensor $\boldsymbol{\Omega}$ is defined by

$$\Omega_{ij} = \rho_s (R_i W_{sj} - R_j W_{si}) \quad (92)$$

is the superfluid vorticity that is directly related to the small-scale state variable \mathbf{w} (by $\boldsymbol{\omega} = -\text{curl} \mathbf{w}$) used in the HVBK model in Sect. 4. Equations (91) and (92) thus relate one of the small-scale fields arising in the two-point extension of the Landau-Tisza model (namely the field W_s) to the small-scale state variable used in the HVBK model. Can we reduce, by using the projection $W_s \rightarrow \mathbf{w}$ defined in (91) and (92), the governing equations (77)–(90) of the two-fluid model to the governing equations of the HVBK model, or the Vinen model (in which $|\text{curl} \mathbf{w}|$ serves as the small-scale variable)? Moreover, keeping the spirit of this paper, we require that the reduction preserves the GENERIC structure of (77)–(90). We hope to address this question in a future paper.

6 Concluding Remarks

In the Reynolds modeling of turbulence the classical hydrodynamic fields are first promoted to the status of random variables and the classical hydrodynamic equations to stochastic equations. This extension is then followed by a reduction to classical hydrodynamic

equations coupled to equations governing the time evolution of new fields describing the small-scale structure of turbulence. The classical hydrodynamic fields as well as the new fields arise as certain averages of the random fields. The averaging process requires a closure relation to provide a self contained system of equations. In this paper we follow the same strategy but with the following changes: (i) Instead of classical fluids we consider the ^4He superfluid. The classical hydrodynamic equations constituting the starting point in the Reynolds modeling are replaced by the governing equations of the two-fluid Landau-Tisza model. (ii) The extension is not to random fields but to two-point fields. (iii) The reduction is made in such a way that the GENERIC structure of the original equations (both the Landau-Tisza equations and their two-point extensions) is preserved. This requirement eliminates the need for closure relations.

Both the two-point view of the ^4He superfluid and the GENERIC structure of the time evolution equations are new features that we are introducing in this paper to the modeling of quantum turbulence. The explicitly nonlocal nature of the two-point fields fits, of course, well the nonlocal nature of the small-scale microstructure induced in the turbulent motion. With the two-point fields we are able to investigate the transport in both the position space and scales. The requirement that the time evolution equations possess the GENERIC structure provides a framework for their systematic construction. In this paper we demonstrate this framework in the setting of the classical Landau-Tisza model, the setting of the HVBK model, and the setting of the new two-point extension of the Landau-Tisza model. The modeling proceeds in three stages. State variables are chosen in the first stage, their kinematics, both Hamiltonian and dissipative, is identified in the second stage. The outcome of the second stage are governing equations involving both the state variables and their conjugates. Thermodynamic analysis establishing the relation between state variables and their conjugates then concludes the modeling.

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